

ON SOME VARIATIONAL PROBLEMS SET ON DOMAINS TENDING TO INFINITY

M. CHIPOT, A. MOJSIC, AND P. ROY

ABSTRACT: Let $\Omega_\ell = \ell\omega_1 \times \omega_2$ where $\omega_1 \subset \mathbb{R}^p$ and $\omega_2 \subset \mathbb{R}^{n-p}$ are assumed to be open and bounded. We consider the following minimization problem:

$$E_{\Omega_\ell}(u_\ell) = \min_{u \in W_0^{1,q}(\Omega_\ell)} E_{\Omega_\ell}(u)$$

where $E_{\Omega_\ell}(u) = \int_{\Omega_\ell} F(\nabla u) - fu$, F is a convex function and $f \in L^{q'}(\omega_2)$. We are interested in studying the asymptotic behavior of the solution u_ℓ as ℓ tends to infinity.

1. INTRODUCTION

For $1 \leq p \leq n-1$ an integer, let $\Omega_\ell = \ell\omega_1 \times \omega_2 \subset \mathbb{R}^n$ where $\omega_1 \subset \mathbb{R}^p$ and $\omega_2 \subset \mathbb{R}^{n-p}$ are open and bounded. ω_1 is also assumed to be star shaped with respect to the origin. We will refer to Ω_ℓ as a cylindrical domain. Points in Ω_ℓ will be denoted by $X = (X_1, X_2)$ where $X_1 = (x_1, \dots, x_p) \in \ell\omega_1$ and $X_2 = (x_{p+1}, \dots, x_n) \in \omega_2$. Let $W^{-1,q'}(\omega)$ denote the dual space of the usual (cf [15]) Sobolev space $W_0^{1,q}(\omega)$. For $q > 1$, let

$$f \in L^{q'}(\omega_2),$$

where $\frac{1}{q} + \frac{1}{q'} = 1$.

Definition (Uniform convexity of power q -type): We say that a function $G : \mathbb{R}^n \rightarrow \mathbb{R}$ is a “uniformly convex function of power q -type” if there exists a constant $\alpha = \alpha(q)$ such that $\forall \xi, \eta \in \mathbb{R}^n$

$$(1.1) \quad 2G\left(\frac{\xi + \eta}{2}\right) + \alpha|\xi - \eta|^q \leq G(\xi) + G(\eta).$$

There is a large amount of literature available on the study of such class of functions. We refer to [20] and the references there. For $q \geq 2$, the function $G(x) = |x|^q$ belongs to such a class. If $1 < q < 2$ then there does not exist any function satisfying (1.1). This is because for finite convex functions it is known [16] that its second order derivative exists almost everywhere, which is contradictory in this case.

Date: February 10, 2016.

Key words and phrases. Cylinders, Variational Problem, Asymptotic analysis.

Let $F : \mathbb{R}^n \rightarrow \mathbb{R}$ be a “uniformly convex function of power q -type”, satisfying the following growth condition:

$$(1.2) \quad \lambda|\xi|^q \leq F(\xi) \leq \Lambda|\xi|^q, \quad \forall \xi \in \mathbb{R}^n,$$

for some $\lambda, \Lambda > 0$.

We consider the following minimization problem:

$$(1.3) \quad E_{\Omega_\ell}(u_\ell) = \min_{u \in W_0^{1,q}(\Omega_\ell)} E_{\Omega_\ell}(u)$$

where

$$(1.4) \quad E_{\Omega_\ell}(u) := \int_{\Omega_\ell} F(\nabla u) - fu.$$

We refer to [14] for the proof of existence and uniqueness of such u_ℓ . In this article we are mainly interested in studying the asymptotic behavior of u_ℓ as ℓ tends to infinity. We consider the following minimization problem defined on the cross section ω_2 of Ω_ℓ :

$$(1.5) \quad E_{\omega_2}(u_\infty) = \min_{u \in W_0^{1,q}(\omega_2)} E_{\omega_2}(u)$$

where $\forall u \in W_0^{1,q}(\omega_2)$,

$$E_{\omega_2}(u) := \int_{\omega_2} F(0, \nabla_{X_2} u) - fu.$$

In [9], the authors considered the same problem for the particular case of

$$F(\xi) = A\xi \cdot \xi,$$

where

$$A := \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^t & A_{22} \end{pmatrix}$$

is $n \times n$ positive definite matrix and “ \cdot ” denotes usual Euclidean scalar product. A_{11} , A_{12} and A_{22} are respectively $p \times p$, $p \times (n-p)$ and $(n-p) \times (n-p)$ matrices with bounded coefficients. A_{12}^t denotes the transpose of the matrix A_{12} . It is easy to see that, in this case F satisfies (1.2) with $q = 2$. This paper can be considered as the principal incentive for our current work. In the case above the unique minimizer u_ℓ additionally satisfies the following Euler-Lagrange equation:

$$(1.6) \quad \begin{cases} -\operatorname{div}(A\nabla u_\ell) = f & \text{in } \Omega_\ell, \\ u_\ell = 0 & \text{on } \partial\Omega_\ell, \end{cases}$$

where div and ∇ denotes the divergence operator and the gradient in the X variable. We recall their main result. Let div_{X_2} and ∇_{X_2} denotes the divergence operator and the gradient in the X_2 variable. For $x > 0$, throughout this paper $[x]$ will denote the greatest integer less than or equal to x .

Theorem 1.1. [Chipot-Yeressian] *There exists some constants $A, B > 0$, such that*

$$(1.7) \quad \int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - u_\infty)|^2 \leq Ae^{-B\ell}$$

where u_∞ is the solution to the problem

$$\begin{cases} -\operatorname{div}_{X_2}(A_{22}\nabla_{X_2}u_\infty) = f & \text{in } \omega_2, \\ u_\infty = 0 & \text{on } \partial\omega_2. \end{cases}$$

Their proof relies on a suitable choice of test function in the weak formulation of (1.6) and an iteration technique.

In [12], the authors studied similar issues for the case of $F(\xi) = |\xi|^q$ where $q \geq 2$. In their case the minimizer satisfies the following Euler-Lagrange equation:

$$\begin{cases} -\operatorname{div}(|\nabla u_\ell|^{q-2}\nabla u_\ell) = f & \text{in } \Omega_\ell, \\ u_\ell = 0 & \text{on } \partial\Omega_\ell. \end{cases}$$

In this case also they obtained the convergence of u_ℓ toward the solution of an associated problem set on the cross section ω_2 . Their main result is the following.

Theorem 1.2. [Chipot-Xie] *For $q \geq 2$, there exists some constants $A, r > 0$, such that*

$$\int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - u_\infty)|^q \leq A\ell^{-r}$$

where u_∞ is the solution to the problem

$$\begin{cases} -\operatorname{div}_{X_2}(|\nabla_{X_2}u_\infty|^{q-2}\nabla_{X_2}u_\infty) = f & \text{in } \omega_2, \\ u = 0 & \text{on } \partial\omega_2. \end{cases}$$

We emphasize that in our main result (Theorem 1.3) we do not assume any regularity on F (except that the condition (1.2) forces F to be differentiable at 0), and hence our problem (1.3) is purely variational in nature (u_ℓ does not satisfy any Euler type equation). Hence this work is a generalization of the work [9] and [12].

The following relation between the problems (1.3) and (1.5) (for large ℓ) is the main result of this paper.

Theorem 1.3. *Under the assumption $p = 1$, $q \geq 2$, (1.1) and (1.2), if u_ℓ and u_∞ satisfy (1.3) and (1.5) respectively then*

$$\int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - u_\infty)|^q \leq \frac{A}{\ell^{\frac{1}{q-1}}},$$

for some constant $A > 0$ (independent of ℓ).

For the case when $q = 2$, note that the rate of convergence in Theorem 1.1 is exponential, where in Theorem 1.3 it is $O(\ell^{-1})$. But it is possible to recover an exponential rate of convergence under an additional assumption on F , namely

$$(1.8) \quad 2F\left(\frac{\xi + \eta}{2}\right) + \beta|\xi - \eta|^2 \geq F(\xi) + F(\eta) \quad \forall \xi, \eta \in \mathbb{R}^n \quad \text{and for some } \beta \geq \alpha.$$

More precisely in this direction our result is the following.

Theorem 1.4. *Under the assumption (1.1), (1.8) and $q = 2$, if u_ℓ and u_∞ satisfy (1.3) and (1.5) respectively then one has*

$$\int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2 \leq Ae^{-B\ell},$$

for some constant $A, B > 0$ (independent of ℓ).

For $q > 2$, affine functions are the only functions which satisfies (1.8) and hence cannot satisfy (1.1) simultanously. This justifies the condition (1.8). The function $F(\xi) = |\xi|^2$, satisfies (1.8), with an equality sign, for $\beta = \frac{1}{2}$ and hence it also satisfies (1.1). It is important here to mention that (1.1) and (1.8) together imply that $F \in C^1(\mathbb{R}^n)$ (see Lemma 4.1). Hence u_ℓ in this case satisfies the following Euler equation:

$$\begin{cases} -\operatorname{div}(\nabla F(\nabla u_\ell)) = f & \text{in } \Omega_\ell, \\ u_\ell = 0 & \text{on } \partial\Omega_\ell. \end{cases}$$

Then, following a technique of [9] one can obtain a different proof of Theorem (1.4). But the proof that we present here uses only the variational structure of the problem.

From the point of view of applications, many problems of mathematical physics are set on large cylindrical domains. For instance, these are porous media flows in channels, plate theory, elasticity theory, etc. Many problems of the type “ $\ell \rightarrow \infty$ ” were studied in the past. Apart from second order elliptic equations [9] already mentioned, this includes eigenvalue problems, parabolic problems, variational inequalities, Stokes problem, hyperbolic problems and many others. We refer to [2, 4, 5, 6, 7, 8, 10, 12, 13, 18, 19] and the references there for the literature available in this direction.

The work of this paper is organized as follows. In the next section we study two problems similar to (1.3) in dimension 1 to develop a better understanding of this kind of issues in a simpler situation. In section 3 we will present the proof of Theorem 1.3. In section 4, we will give the proof of Theorem 1.4. A partial result is obtained when the cylinder goes to infinity in more than one directions [see, Theorem 4.1]. We will also present some results regarding the asymptotic behavior of $\frac{E_{\Omega_\ell}(u_\ell)}{2\ell}$ as ℓ tends to infinity.

2. A ONE DIMENSIONAL PROBLEM

In this section we assume that $\Omega_\ell = (-\ell, \ell)$, F is a function satisfying (1.2). The functional, analogous to (1.4) in one dimensional situation, is defined as

$$E_{\Omega_\ell}(u) = \int_{-\ell}^{\ell} F(u') - \gamma u, \quad u \in W_0^{1,q}(\Omega_\ell)$$

where $\gamma \in \mathbb{R}$. To observe the asymptotic behavior of the minimizers u_ℓ , we take the test case when $F(x) = \frac{x^2}{2}$. Then one can write down explicitly the solution, after solving the associated Euler-Lagrange equation, as $u_\ell(x) = \frac{\gamma}{2}(\ell^2 - x^2)$, $x \in (-\ell, \ell)$. This implies that $u_\ell \rightarrow \pm\infty$ pointwise depending on the sign of γ .

Now we consider the case of a general F satisfying (1.2) and $\gamma > 0$ (we are restricting ourselves to a constant γ but the proof goes through for a positive function γ bounded away from zero uniformly). We do not assume any convexity on F but the existence of a minimizer u_ℓ that we consider next. First note that

$$u_\ell \geq 0, \quad \text{in } \Omega_\ell.$$

This is because, if $u_\ell < 0$ on $A \subset \Omega_\ell$, where $|A| > 0$. Then defining $w_\ell = \max\{0, u_\ell\}$, one gets $E_{\Omega_\ell}(w_\ell) < E_{\Omega_\ell}(u_\ell)$, which contradicts the definition of u_ℓ .

We define

$$u_\ell(x_\ell) = \max_{x \in [-\ell, \ell]} u_\ell(x).$$

Note that since u_ℓ is a continuous function, such a $x_\ell \in [-\ell, \ell]$ always exists.

Lemma 2.1. *u_ℓ is non decreasing on $(-\ell, x_\ell)$ and non increasing on (x_ℓ, ℓ) .*

Proof. If u_ℓ is not non decreasing on $(-\ell, x_\ell)$ there exist y_0 and y_1 such that

$$-\ell < y_0 < y_1 < x_\ell, \quad u_\ell(y_0) > u_\ell(y_1).$$

Let z be a point in (y_0, x_ℓ) such that $u_\ell(z) = \min_{(y_0, x_\ell)} u_\ell$ and I_z the connected component of $\{x \mid u_\ell(x) < u_\ell(y_0)\}$ containing z . Define then ϕ_ℓ by

$$\phi_\ell = \begin{cases} u_\ell & \text{on } \Omega_\ell \setminus I_z, \\ u_\ell(y_0) & \text{on } I_z. \end{cases}$$

Clearly this function satisfies $E_{\Omega_\ell}(\phi_\ell) < E_{\Omega_\ell}(u_\ell)$ which contradicts the definition of u_ℓ . The proof that u_ℓ is non increasing on (x_ℓ, ℓ) follows the same way. \square

Theorem 2.1. *$u_\ell(x) \rightarrow \infty$ pointwise $\forall x$, and for fixed $a < b$, one has $\int_a^b u_\ell^s \rightarrow \infty$, for all $s > 0$.*

Proof. Let us argue by contradiction. Suppose that there exist $K > 0$, $z \in \mathbb{R}$ and a sequence $l_k \rightarrow \infty$ (which is again labeled by $\{\ell\}$), such that

$0 \leq u_\ell(z) \leq K$. Let us assume that $x_\ell \leq z$ for all ℓ . This implies from the previous lemma that $u_\ell(x) \leq K$, $x \geq z$. Let $\delta > 0$, define the function

$$\phi_\ell(x) = \begin{cases} u_\ell & \text{on } I_1 := (-\ell, z], \\ u_\ell(z) + (x - z)(u_\ell(z + 1) - u_\ell(z) + \delta) & \text{on } I_2 := (z, z + 1], \\ u_\ell(x) + \delta & \text{on } I_3 := [z + 1, \ell - 1), \\ (\ell - x)(u_\ell(\ell - 1) + \delta) & \text{on } I_4 := (\ell - 1, \ell]. \end{cases}$$

It is easy to see that

$$(2.1) \quad E_{I_1}(u_\ell) - E_{I_1}(\phi_\ell) = 0, \quad E_{I_3}(\phi_\ell) - E_{I_3}(u_\ell) \leq -\gamma\delta(\ell - z - 2).$$

Now using the fact that $0 \leq \phi_\ell \leq K + \delta$ on $I_2 \cup I_4$, it is possible to find a constant $C > 0$ (independent of ℓ) such that

$$(2.2) \quad E_{I_2 \cup I_4}(\phi_\ell) - E_{I_2 \cup I_4}(u_\ell) \leq C.$$

Adding (2.1) and (2.2), we obtain, when ℓ is sufficiently large

$$E_{\Omega_\ell}(\phi_\ell) - E_{\Omega_\ell}(u_\ell) \leq C - \gamma\delta(\ell - z - 2) < 0,$$

which is a contradiction to the definition of u_ℓ . If $x_\ell \geq z$ for some ℓ large defining ϕ_ℓ analogously on the other side of x_ℓ we arrive to the same contradiction and the proof follows.

Then, using the monotonicity property of u_ℓ , one has

$$\int_a^b u_\ell^s \geq \min\{u_\ell(a), u_\ell(b)\}^s (b - a) \rightarrow \infty.$$

This completes the proof of the theorem. \square

The situation can be different if $\gamma = 0$ and if some coerciveness is added (compare to [11]). Suppose $\alpha, \beta > 0$, define the energy functional, $E_{\Omega_\ell} : W_{\alpha, \beta}^{1, q}(\Omega_\ell) \rightarrow \mathbb{R}$ as

$$E_{\Omega_\ell}(u) = \int_{-\ell}^{\ell} F(u') + |u|^q$$

where $W_{\alpha, \beta}^{1, q}(\Omega_\ell) := \{u \in W^{1, q}(\Omega_\ell) \mid u(-\ell) = \alpha, u(\ell) = \beta\}$. Consider then the following minimization problem:

$$(2.3) \quad E_{\Omega_\ell}(v_\ell) = \inf_{u \in W_{\alpha, \beta}^{1, q}(\Omega_\ell)} E_{\Omega_\ell}(u).$$

For existence and uniqueness of such function v_ℓ , we refer to [14] but we just assume here existence of some minimizer v_ℓ that we consider next.

Lemma 2.2. *It holds that*

$$0 \leq v_\ell \leq \max\{\alpha, \beta\} \quad \text{and} \quad \int_{-\ell}^{\ell} |v'_\ell|^q + v_\ell^q \leq C$$

where C is a positive constant, independent of ℓ .

Proof. Let $|\{v_\ell < 0\}| > 0$. Define $w_\ell = \max\{0, v_\ell\}$. Then $E_{\Omega_\ell}(w_\ell) < E_{\Omega_\ell}(v_\ell)$, which contradicts the definition of v_ℓ . The fact that $v_\ell \leq \max\{\alpha, \beta\}$ follows with a similar argument, with the function $W_\ell = \min\{v_\ell, \max\{\alpha, \beta\}\}$ playing the role of w_ℓ in the previous part.

Clearly the function

$$\phi_\ell(x_1) = \begin{cases} -\alpha x_1 + \alpha - \alpha\ell & \text{on } (-\ell, -\ell + 1], \\ 0 & \text{on } (-\ell + 1, \ell - 1), \\ \beta x_1 + \beta - \ell\beta & \text{on } [\ell - 1, \ell), \end{cases}$$

is in the space $W_{\alpha, \beta}^{1, q}(\Omega_\ell)$. Then it holds also that for all $\ell \geq 1$,

$$E_{\Omega_\ell}(\phi_\ell) = E_{\Omega_1}(\phi_1).$$

Using ϕ_ℓ as test function in (2.3), one gets

$$E_{\Omega_\ell}(v_\ell) \leq E_{\Omega_\ell}(\phi_\ell) = E_{\Omega_1}(\phi_1).$$

This proves the lemma. \square

Next we state the main theorem of this section which shows exponential rate of convergence of the solutions v_ℓ towards 0. The method of proof depends on an iteration scheme.

Theorem 2.2. *It holds that*

$$\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} |v'_\ell|^q + v_\ell^q \leq C e^{-\alpha\ell},$$

where C and α are some positive constants, independent of ℓ .

Proof. Suppose $\ell_1 \leq \ell$. Consider the following test function ϕ_{ℓ_1} defined on $\Omega_\ell = (-\ell, \ell)$ by

$$\phi_{\ell_1}(x_1) = \begin{cases} 1 & \text{on } [-\ell, -\ell_1] \cup [\ell_1, \ell], \\ -x_1 - \ell_1 + 1 & \text{on } (-\ell_1, -\ell_1 + 1), \\ x_1 - \ell_1 + 1 & \text{on } (\ell_1 - 1, \ell_1), \\ 0 & \text{on } [-\ell_1 + 1, \ell_1 - 1]. \end{cases}$$

The graph of ϕ_{ℓ_1} is shown below.

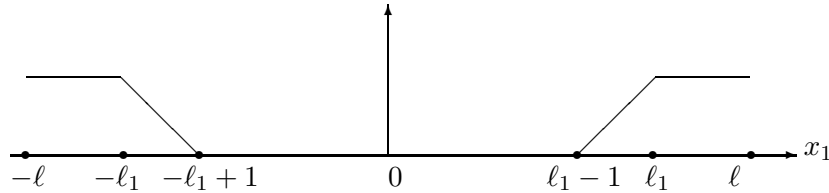


figure 1

Clearly we have $\phi_{\ell_1} v_\ell \in W_{\alpha, \beta}^{1,q}(\Omega_\ell)$. Hence from (2.3), we get $E_{\Omega_\ell}(v_\ell) \leq E_{\Omega_\ell}(\phi_{\ell_1} v_\ell)$. Since $\phi_{\ell_1} = 1$ on the set $\Omega_\ell \setminus \Omega_{\ell_1}$, we have

$$\int_{\Omega_{\ell_1}} F(v'_\ell) + v_\ell^q \leq \int_{\Omega_{\ell_1}} F((\phi_{\ell_1} v_\ell)') + (\phi_{\ell_1} v_\ell)^q.$$

Setting $D_{\ell_1} := \Omega_{\ell_1} \setminus \Omega_{\ell_1-1}$ and observing that $\phi_{\ell_1} = 0$ on Ω_{ℓ_1-1} , we have

$$\int_{\Omega_{\ell_1}} F(v'_\ell) + v_\ell^q \leq \int_{D_{\ell_1}} F((\phi_{\ell_1} v_\ell)') + (\phi_{\ell_1} v_\ell)^q.$$

Using (1.2), convexity of the function x^q and properties of ϕ_{ℓ_1} we have for some constant $C > 0$ (independent of ℓ)

$$\begin{aligned} \min\{\lambda, 1\} \left(\int_{\Omega_{\ell_1}} |v'_\ell|^q + v_\ell^q \right) &\leq \Lambda \int_{D_{\ell_1}} |(\phi_{\ell_1} v_\ell)'|^q + \int_{D_{\ell_1}} (\phi_{\ell_1} v_\ell)^q \\ &\leq \Lambda \int_{D_{\ell_1}} |\phi_{\ell_1} v'_\ell + v_\ell \phi'_{\ell_1}|^q + \int_{D_{\ell_1}} v_\ell^q \\ &\leq C \int_{D_{\ell_1}} |v'_\ell|^q + v_\ell^q. \end{aligned}$$

The above inequality implies for $\tilde{\Lambda} = \frac{C}{\min\{\lambda, 1\}}$

$$\int_{\Omega_{\ell_1-1}} |v'_\ell|^q + v_\ell^q \leq \frac{\tilde{\Lambda}}{\tilde{\Lambda} + 1} \left(\int_{\Omega_{\ell_1}} |v'_\ell|^q + v_\ell^q \right).$$

Now choosing $\ell_1 = \frac{\ell}{2} + 1, \frac{\ell}{2} + 2, \dots, \frac{\ell}{2} + [\frac{\ell}{2}]$ and iterating the above formula, we obtain

$$\int_{\Omega_{\frac{\ell}{2}}} |v'_\ell|^q + v_\ell^q \leq \left(\frac{\tilde{\Lambda}}{1 + \tilde{\Lambda}} \right)^{[\frac{\ell}{2}]} \left(\int_{\Omega_{\frac{\ell}{2} + [\frac{\ell}{2}]}} |v'_\ell|^q + v_\ell^q \right).$$

since $\frac{\ell}{2} - 1 < [\frac{\ell}{2}] \leq \frac{\ell}{2}$ there are positive constants C_1 and C_2 such that

$$\int_{\Omega_{\frac{\ell}{2}}} |v'_\ell|^q + v_\ell^q \leq C_1 e^{-C_2 \ell} \left(\int_{\Omega_\ell} |v'_\ell|^q + v_\ell^q \right).$$

The theorem then follows from the last lemma. \square

3. PROOF OF THEOREM 1.3

We define the spaces

$$W_{lat}^{1,q}(\Omega_\ell) := \{u \in W^{1,q}(\Omega_\ell) \mid u = 0 \text{ on } \ell\omega_1 \times \partial\omega_2\}$$

and

$$V^{1,q}(\Omega_\ell) = \left\{ u \in W_{lat}^{1,q}(\Omega_\ell) \mid \begin{array}{l} u(Y, X_2) = u(Z, X_2), \\ a.e. (Y, X_2), (Z, X_2) \in \partial(\ell\omega_1) \times \omega_2 \end{array} \right\}.$$

We consider the following minimization problem on $V^{1,q}(\Omega_\ell)$:

$$(3.1) \quad E_{\Omega_\ell}(w_\ell) = \min_{u \in V^{1,q}(\Omega_\ell)} E_{\Omega_\ell}(u),$$

where E_{Ω_ℓ} is defined as in (1.3). Existence and uniqueness of such w_ℓ again follows as for the case of Problem (1.3), assuming (1.2) and F strictly convex.

Theorem 3.1. *One has $\forall \ell$, $w_\ell = u_\infty$, where u_∞ is as in (1.5).*

Proof. Let $u \in V^{1,q}(\Omega_\ell)$. Set

$$v(X_2) = \int_{\ell\omega_1} u(\cdot, X_2) := \frac{1}{|\ell\omega_1|} \int_{\ell\omega_1} u(\cdot, X_2)$$

where $|\ell\omega_1|$ denotes the measure of $\ell\omega_1$. It is easy to see that $v \in W_0^{1,q}(\omega_2)$. Thus

$$E_{\omega_2}(u_\infty) \leq E_{\omega_2}(v) = \int_{\omega_2} F(0, \nabla_{X_2} v) - f v.$$

Now using the fact that $u(\cdot, X_2)$ is constant on the set $\partial(\ell\omega_1)$ and from the divergence theorem, one has

$$0 = \int_{\ell\omega_1} \nabla_{X_1} u.$$

Also by differentiation under the integral, we get

$$\nabla_{X_2} v = \int_{\ell\omega_1} \nabla_{X_2} u.$$

Therefore we have by Jensen's inequality

$$\begin{aligned} E_{\omega_2}(u_\infty) &\leq \int_{\omega_2} \left\{ F \left(\int_{\ell\omega_1} \nabla_{X_1} u, \int_{\ell\omega_1} \nabla_{X_2} u \right) \right\} - \int_{\omega_2} f \left\{ \int_{\ell\omega_1} u \right\} \\ &\leq \int_{\omega_2} \int_{\ell\omega_1} \{ F(\nabla u) - f u \} = \frac{E_{\Omega_\ell}(u)}{|\ell\omega_1|}. \end{aligned}$$

Clearly equality holds in the above inequality if $u = u_\infty$. Then the claim follows from the uniqueness of the minimizer. \square

Next we consider the case when $p = 1$, which means that the cylinder Ω_ℓ becomes unbounded in one direction. Points in ω_1 are now simply denoted by x_1 . ∇, ∇_{X_2} denote the gradients in X and X_2 variables respectively. By

$$|u|_{q,\Omega} = \left(\int_{\Omega} |u|^q \right)^{\frac{1}{q}},$$

we will denote the L^q norm of u on any domain Ω .

We need a few preliminaries before proceeding to the proof of Theorem 1.3. We know that a convex function is locally Lipschitz continuous on \mathbb{R}^n , the next lemma provides an estimate on the growth of the Lipschitz constant for convex functions satisfying (1.2).

Proposition 3.1. For convex function $F : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying (1.2), for every $P, Q \in \mathbb{R}^n$, we have

$$(3.2) \quad |F(Q) - F(P)| \leq 2^q \Lambda \max \left\{ |P|^{q-1}, |Q|^{q-1} \right\} |Q - P| \\ \leq 2^q \Lambda (|P|^{q-1} + |Q|^{q-1}) |P - Q|.$$

Proof. First note that the second inequality holds trivially. For a convex function $f : \mathbb{R} \rightarrow \mathbb{R}$ and $a < b < c$, we know that

$$(3.3) \quad \frac{f(b) - f(a)}{b - a} \leq \frac{f(c) - f(b)}{c - b}.$$

The function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$f(t) = F((1-t)P + tQ),$$

is clearly convex. Using the inequality (3.3) for $a = 0$, $b = 1$ and $c = t > 1$ we have

$$F(Q) - F(P) \leq \frac{F((1-t)P + tQ) - F(Q)}{t - 1} \leq \frac{\Lambda |(1-t)P + tQ|^q}{t - 1}.$$

Now for $t = 1 + \frac{|Q|}{|Q-P|}$, we have

$$\begin{aligned} F(Q) - F(P) &\leq \frac{\Lambda \left| Q + \frac{|Q|}{|Q-P|} (Q - P) \right|^q}{|Q|} |Q - P| \\ &\leq \frac{\Lambda \left(|Q| + \frac{|Q|}{|Q-P|} |Q - P| \right)^q}{|Q|} |Q - P| \\ &\leq 2^q \Lambda |Q|^{q-1} |Q - P|. \end{aligned}$$

Now changing the roles of P and Q , we have

$$F(P) - F(Q) \leq 2^q \Lambda |P|^{q-1} |P - Q|,$$

and (3.2) follows. \square

The following result is well known [2], we state it without proof.

Lemma 3.1. [The Poincaré's Inequality for ω_2] Let $0 < s < t$ and $q \geq 1$, then there exists $\lambda_1 = \lambda_1(q, \omega_2) > 0$ (independent of s and t) such that

$$|u|_{q, (s,t) \times \omega_2} \leq \lambda_1 \|\nabla u\|_{q, (s,t) \times \omega_2},$$

for all $u \in W^{1,q}((s,t) \times \omega_2)$ with $u = 0$ on $(s,t) \times \partial\omega_2$.

Now our main aim is to prove the Corollary 3.1 below, which we will use in the proof of Theorem 1.3.

For $s < t$, $\rho_{s,t} = \rho_{s,t}(x_1)$ denotes a real, Lipschitz continuous function such that

$$(3.4) \quad 0 \leq \rho_{s,t} \leq 1, \quad |\rho'_{s,t}| \leq C, \quad \rho_{s,t} = 1 \text{ on } (s,t), \quad \rho_{s,t} = 0 \text{ outside } (s-1, t+1)$$

and $D_{s,t}$ denotes the set defined as

$$D_{s,t} = (((s-1, t+1) \times \omega_2) \setminus ((s, t) \times \omega_2)) \cap \Omega_\ell.$$

Proposition 3.2. *Let $p = 1$. There exists a constant (independent of ℓ) such that for $-\ell < s < t < \ell$ we have*

$$E_{(s,t) \times \omega_2}(u_\ell) \leq C|f|_{q', \omega_2}^{\frac{q}{q-1}}.$$

In particular it implies that for $\ell_0 < \ell$,

$$(3.5) \quad E_{D_{\ell_0}}(u_\ell) \leq 2C|f|_{q', \omega_2}^{\frac{q}{q-1}}$$

where $D_{\ell_0} := \Omega_\ell \cap (\Omega_{\ell_0+1} \setminus \Omega_{\ell_0})$ and u_ℓ is as in (1.3).

Proof. First we claim that if $E_{D_{s,t}}(u_\ell) \leq 0$ then

$$(3.6) \quad E_{(s,t) \times \omega_2}(u_\ell) \leq C(\lambda, \lambda_1, \Lambda)|f|_{q', \omega_2}^{\frac{q}{q-1}}.$$

Indeed if $E_{D_{s,t}}(u_\ell) \leq 0$ then one has

$$\begin{aligned} \lambda \int_{D_{s,t}} |\nabla u_\ell|^q &\leq \int_{D_{s,t}} F(\nabla u_\ell) \leq \int_{D_{s,t}} f u_\ell \leq |f|_{q', D_{s,t}} |u_\ell|_{q, D_{s,t}} \\ &\leq \lambda_1 |f|_{q', D_{s,t}} \|\nabla u_\ell\|_{q, D_{s,t}} \end{aligned}$$

by the Poincaré inequality for ω_2 . It follows that

$$(3.7) \quad \|\nabla u_\ell\|_{q, D_{s,t}}^q \leq \left(\frac{\lambda_1}{\lambda}\right)^{\frac{q}{q-1}} |f|_{q', D_{s,t}}^{\frac{q}{q-1}}.$$

Since $(1 - \rho_{s,t})u_\ell \in W_0^{1,q}(\Omega_\ell)$ we have

$$E_{\Omega_\ell}(u_\ell) \leq E_{\Omega_\ell}((1 - \rho_{s,t})u_\ell),$$

that is to say

$$\begin{aligned} E_{(s,t) \times \omega_2}(u_\ell) + E_{D_{s,t}}(u_\ell) + E_{\Omega_\ell \setminus (s-1, t+1) \times \omega_2}(u_\ell) &\leq E_{(s,t) \times \omega_2}((1 - \rho_{s,t})u_\ell) \\ &\quad + E_{D_{s,t}}((1 - \rho_{s,t})u_\ell) + E_{\Omega_\ell \setminus (s-1, t+1) \times \omega_2}((1 - \rho_{s,t})u_\ell). \end{aligned}$$

Since $\rho_{s,t} = 0$ outside $(s-1, t+1)$ $\rho_{s,t} = 1$ on (s, t) we get

$$\begin{aligned} E_{(s,t) \times \omega_2}(u_\ell) &\leq E_{D_{s,t}}((1 - \rho_{s,t})u_\ell) - E_{D_{s,t}}(u_\ell) \\ &= \int_{D_{s,t}} F(\nabla(1 - \rho_{s,t})u_\ell) - F(\nabla u_\ell) + f \rho_{s,t} u_\ell \\ &\leq \Lambda \left\{ \|\nabla(1 - \rho_{s,t})u_\ell\|_{q, D_{s,t}}^q + \|\nabla u_\ell\|_{q, D_{s,t}}^q \right\} + |f|_{q', D_{s,t}} |u_\ell|_{q, D_{s,t}} \\ &\leq \Lambda \left\{ \|(1 - \rho_{s,t})\nabla u_\ell - u_\ell \nabla \rho_{s,t}\|_{q, D_{s,t}}^q + \|\nabla u_\ell\|_{q, D_{s,t}}^q \right\} \\ &\quad + \lambda_1 |f|_{q', D_{s,t}} \|\nabla u_\ell\|_{q, D_{s,t}}. \end{aligned}$$

Now using the triangle inequality and Poincaré inequality for ω_2 we obtain for some constants $K(q)$ and $C(\lambda, \lambda_1, \Lambda, q)$,

$$\begin{aligned} E_{(s,t) \times \omega_2}(u_\ell) &\leq \Lambda K \left\{ |u_\ell|_{q, D_{s,t}}^q + \|\nabla u_\ell\|_{q, D_{s,t}}^q \right\} + \lambda_1 |f|_{q', D_{s,t}} |\nabla u_\ell|_{q, D_{s,t}} \\ &\leq C(\lambda, \lambda_1, \Lambda, q) |f|_{q', \omega_2}^{\frac{q}{q-1}} \end{aligned}$$

by (3.7). This completes the claim i.e. shows (3.6).

Next we can complete the proof of the theorem. Indeed, let m be the first nonnegative integer such that $D_{s-m, t+m}$ is non empty and

$$E_{D_{s-m, t+m}}(u_\ell) \leq 0.$$

If there is no such integer, then

$$E_{(s,t) \times \omega_2}(u_\ell) \leq E_{(s-1, t+1) \times \omega_2}(u_\ell) \leq \dots \leq E_{\Omega_\ell}(u_\ell) \leq 0.$$

In the last inequality we used the fact that

$$E_{\Omega_\ell}(u_\ell) \leq E_{\Omega_\ell}(0) = 0.$$

If such an m exists then, by the first part of this proof,

$$E_{(s,t) \times \omega_2}(u_\ell) \leq E_{(s-m, t+m) \times \omega_2}(u_\ell) \leq C |f|_{q', \omega_2}^{\frac{q}{q-1}}.$$

This proves the proposition. \square

Corollary 3.1. [Gradient Estimate] *For every $\ell_0 \leq \ell$ we have,*

$$\int_{D_{\ell_0}} |\nabla u_\ell|^q \leq \tilde{C},$$

for some constant $\tilde{C} = \tilde{C}(\lambda, \lambda_1, \Lambda, f)$

Proof. Applying Hölder's, Young's inequality and (1.2), to (3.5), we obtain

$$\begin{aligned} \lambda \int_{D_{\ell_0}} |\nabla u_\ell|^q &\leq \int_{D_{\ell_0}} F(\nabla u_\ell) \\ &\leq 2C |f|_{q', \omega_2}^{q'} + \int_{D_{\ell_0}} f u_\ell \\ &\leq 2C |f|_{q', \omega_2}^{q'} + \frac{1}{\epsilon^{q'} q'} |f|_{q', D_{\ell_0}}^{q'} + \frac{\epsilon^q}{q} |u_\ell|_{q, D_{\ell_0}}^q \\ &\leq 2C |f|_{q', \omega_2}^{q'} + \frac{1}{\epsilon^{q'} q'} |f|_{q', D_{\ell_0}}^{q'} + \frac{\lambda_1^q \epsilon^q}{q} \|\nabla u_\ell\|_{q, D_{\ell_0}}^q. \end{aligned}$$

This implies that

$$\left(\lambda - \frac{\lambda_1^q \epsilon^q}{q} \right) \int_{D_{\ell_0}} |\nabla u_\ell|^q \leq 2C |f|_{q', \omega_2}^{q'} + \frac{1}{\epsilon^{q'} q'} |f|_{q', D_{\ell_0}}^{q'}.$$

The corollary follows after choosing ϵ such that $\lambda - \frac{\lambda_1^q \epsilon^q}{q} = \frac{1}{2}$. \square

Proof of Theorem 1.3 For some $0 \leq \ell_0 \leq \ell - 1$, we define the functions

$$(3.8) \quad \begin{aligned} \hat{\psi}_{\ell_0, \ell} &= u_\ell + \frac{\rho_{\ell_0}}{2} (u_\infty - u_\ell) \text{ and} \\ \check{\psi}_{\ell_0, \ell} &= u_\infty + \frac{\rho_{\ell_0}}{2} (u_\ell - u_\infty), \end{aligned}$$

where ρ_{ℓ_0} is defined before in (3.4) with $s = -\ell_0$ and $t = \ell_0$. Because $\hat{\psi}_{\ell_0, \ell} \in W_0^{1,q}(\Omega_\ell)$, we have from (1.3)

$$(3.9) \quad E_{\Omega_\ell}(u_\ell) \leq E_{\Omega_\ell}(\hat{\psi}_{\ell_0, \ell}).$$

Also since $\check{\psi}_{\ell_0, \ell} \in V^{1,q}(\Omega_\ell)$, from Theorem 3.1

$$(3.10) \quad E_{\Omega_\ell}(u_\infty) \leq E_{\Omega_\ell}(\check{\psi}_{\ell_0, \ell}).$$

From (3.8) it is easy to see that

$$(3.11) \quad \hat{\psi}_{\ell_0, \ell} + \check{\psi}_{\ell_0, \ell} = u_\ell + u_\infty.$$

Adding (3.9) and (3.10), we have

$$E_{\Omega_\ell}(u_\ell) + E_{\Omega_\ell}(u_\infty) \leq E_{\Omega_\ell}(\nabla \hat{\psi}_{\ell_0, \ell}) + E_{\Omega_\ell}(\nabla \check{\psi}_{\ell_0, \ell})$$

which implies

$$\int_{\Omega_\ell} F(\nabla u_\ell) + F(\nabla u_\infty) \leq \int_{\Omega_\ell} F(\hat{\psi}_{\ell_0, \ell}) + F(\check{\psi}_{\ell_0, \ell}).$$

Using the fact that $u_\ell = \hat{\psi}_{\ell_0, \ell}$ and $u_\infty = \check{\psi}_{\ell_0, \ell}$ on $\Omega_\ell \setminus \Omega_{\ell_0+1}$, we get

$$\int_{\Omega_{\ell_0+1}} F(\nabla u_\ell) + F(\nabla u_\infty) \leq \int_{\Omega_{\ell_0+1}} F(\nabla \hat{\psi}_{\ell_0, \ell}) + F(\nabla \check{\psi}_{\ell_0, \ell}).$$

Note that $\hat{\psi}_{\ell_0, \ell} = \check{\psi}_{\ell_0, \ell} = \frac{u_\ell + u_\infty}{2}$ on Ω_{ℓ_0} , which gives

$$(3.12) \quad \begin{aligned} \int_{\Omega_{\ell_0}} F(\nabla u_\ell) + F(\nabla u_\infty) - 2F\left(\frac{\nabla u_\ell + \nabla u_\infty}{2}\right) \\ \leq \int_{D_{\ell_0}} F(\nabla \hat{\psi}_{\ell_0, \ell}) + F(\nabla \check{\psi}_{\ell_0, \ell}) - F(\nabla u_\ell) - F(\nabla u_\infty) := I. \end{aligned}$$

Using the convexity of F and (3.11), we have

$$\begin{aligned} I &= \int_{D_{\ell_0}} F(\nabla \hat{\psi}_{\ell_0, \ell}) + F(\nabla \check{\psi}_{\ell_0, \ell}) - F(\nabla u_\ell) - F(\nabla u_\infty) \\ &\leq \int_{D_{\ell_0}} F(\nabla \hat{\psi}_{\ell_0, \ell}) + F(\nabla \check{\psi}_{\ell_0, \ell}) - 2F\left(\frac{\nabla u_\ell + \nabla u_\infty}{2}\right) \\ &= \int_{D_{\ell_0}} F(\nabla \hat{\psi}_{\ell_0, \ell}) + F(\nabla \check{\psi}_{\ell_0, \ell}) - 2F\left(\frac{\nabla \hat{\psi}_{\ell_0, \ell} + \nabla \check{\psi}_{\ell_0, \ell}}{2}\right). \end{aligned}$$

Now using Proposition 3.1, we obtain for some constant $C = C(q, \Lambda) > 0$,

(3.13)

$$|I| \leq C \int_{D_{\ell_0}} \left\{ |\nabla \hat{\psi}_{\ell_0, \ell}|^{q-1} + |\nabla(\hat{\psi}_{\ell_0, \ell} + \check{\psi}_{\ell_0, \ell})|^{q-1} + |\nabla \check{\psi}_{\ell_0, \ell}|^{q-1} \right\} |\nabla(\hat{\psi}_{\ell_0, \ell} - \check{\psi}_{\ell_0, \ell})|.$$

Since $q \geq 2$, using the monotonicity of the function $|X|^{q-1}$, we have for $a, b > 0$

$$(a + b)^{q-1} \leq (2 \max\{a, b\})^{q-1} = 2^{q-1} \max\{a, b\}^{q-1} \leq 2^{q-1} (a^{q-1} + b^{q-1}).$$

Now from (3.13) we get,

$$I \leq C \int_{D_{\ell_0}} \left(|\nabla \hat{\psi}_{\ell_0, \ell}|^{q-1} + |\nabla \check{\psi}_{\ell_0, \ell}|^{q-1} \right) |\nabla(\hat{\psi}_{\ell_0, \ell} - \check{\psi}_{\ell_0, \ell})|.$$

Then using Hölder's inequality, we obtain

$$(3.14) \quad I \leq C \left(\|\nabla \hat{\psi}_{\ell_0, \ell}\|_{q, D_{\ell_0}}^{\frac{q}{q-1}} + \|\nabla \check{\psi}_{\ell_0, \ell}\|_{q, D_{\ell_0}}^{\frac{q}{q-1}} \right) \|\nabla(\hat{\psi}_{\ell_0, \ell} - \check{\psi}_{\ell_0, \ell})\|_{q, D_{\ell_0}}.$$

Since F is uniformly convex of power q -type, we get

$$(3.15) \quad \alpha \int_{\Omega_{\ell_0}} |\nabla(u_\ell - u_\infty)|^q \leq \int_{\Omega_{\ell_0}} F(\nabla u_\ell) + F(\nabla u_\infty) - 2F\left(\frac{\nabla u_\ell + \nabla u_\infty}{2}\right).$$

Combining (3.12), (3.14) and (3.15), one gets

$$(3.16) \quad \alpha \int_{\Omega_{\ell_0}} |\nabla(u_\ell - u_\infty)|^q \leq C \left(\|\nabla \hat{\psi}_{\ell_0, \ell}\|_{q, D_{\ell_0}}^{\frac{q}{q-1}} + \|\nabla \check{\psi}_{\ell_0, \ell}\|_{q, D_{\ell_0}}^{\frac{q}{q-1}} \right) \|\nabla(\hat{\psi}_{\ell_0, \ell} - \check{\psi}_{\ell_0, \ell})\|_{q, D_{\ell_0}}.$$

We estimate each integral on the right hand side of the above inequality. One has for some constant $C > 0$,

$$\begin{aligned} \int_{D_{\ell_0}} |\nabla(\hat{\psi}_{\ell_0, \ell} - \check{\psi}_{\ell_0, \ell})|^q &= \int_{D_{\ell_0}} |\nabla\{(1 - \rho_{\ell_0})(u_\ell - u_\infty)\}|^q \\ &\leq 2^{q-1} \int_{D_{\ell_0}} (1 - \rho_{\ell_0})^q |\nabla(u_\ell - u_\infty)|^q + 2^{q-1} \int_{D_{\ell_0}} (u_\ell - u_\infty)^q |\nabla \rho_{\ell_0}|^q. \end{aligned}$$

Using Poincaré's inequality and the properties of ρ_{ℓ_0} , one has for some $K = K(\lambda_1)$

$$(3.17) \quad \int_{D_{\ell_0}} |\nabla(\hat{\psi}_{\ell_0, \ell} - \check{\psi}_{\ell_0, \ell})|^q \leq K \int_{D_{\ell_0}} |\nabla(u_\ell - u_\infty)|^q.$$

Applying Corollary 3.1, one can estimate the terms $\|\nabla \hat{\psi}_{\ell_0, \ell}\|_{q, D_{\ell_0}}$ and $\|\nabla \check{\psi}_{\ell_0, \ell}\|_{q, D_{\ell_0}}$ similarly, to obtain for some constant $K_1 = K_1(f, \lambda_1)$

$$\int_{D_{\ell_0}} |\nabla \hat{\psi}_{\ell_0, \ell}|^q, \int_{D_{\ell_0}} |\nabla \check{\psi}_{\ell_0, \ell}|^q \leq K_1.$$

For some other constant M (which depends only on K, K_1), (3.16) becomes

$$(3.18) \quad \int_{\Omega_{\ell_0}} |\nabla(u_\ell - u_\infty)|^q \leq M \left(\int_{D_{\ell_0}} |\nabla(u_\ell - u_\infty)|^q \right)^{\frac{1}{q}}.$$

Applying Corollary 3.1, we get for some other constant M_1

$$(3.19) \quad \int_{\Omega_{\ell_0}} |\nabla(u_\ell - u_\infty)|^q \leq M_1.$$

Denoting

$$a_m = \int_{\Omega_{\frac{\ell}{2}+m}} |\nabla(u_\ell - u_\infty)|^q$$

by (3.18) we have for $m = 0, \dots, [\frac{\ell}{2}] - 1$

$$(3.20) \quad a_m \leq M(a_{m+1} - a_m)^{\frac{1}{q}}.$$

One may see that there exists $t_0 > 1$ such that for $1 < t < t_0$ we have

$$\frac{1}{t^{q-1}} \leq 1 - \frac{1}{2}(q-1)(t-1).$$

It follows that by taking $t = \frac{a_{m+1}}{a_m}$ we have that if $\frac{a_{m+1}}{a_m} < t_0$ then

$$(3.21) \quad (q-1) \frac{a_{m+1} - a_m}{a_m^q} \leq 2(a_m^{1-q} - a_{m+1}^{1-q}).$$

Thus in the case $\frac{a_{m+1}}{a_m} < t_0$ by (3.20) and (3.21) we have

$$(3.22) \quad M^{-q} \leq \frac{a_{m+1} - a_m}{a_m^q} \leq \frac{2}{q-1}(a_m^{1-q} - a_{m+1}^{1-q}).$$

In the case $\frac{a_{m+1}}{a_m} > t_0$, using the bound $a_m < M_1$ we compute

$$a_m^{1-q} - a_{m+1}^{1-q} \geq a_m^{1-q}(1 - t_0^{1-q}) = M_1(1 - t_0^{1-q}).$$

Thus we have

$$a_m^{1-q} - a_{m+1}^{1-q} \geq \min \left\{ M^{-q} \frac{q-1}{2}, M_1^{1-q}(1 - t_0^{1-q}) \right\} = C_1.$$

Summing this inequality for $m = 0, \dots, [\frac{\ell}{2}] - 1$ we obtain

$$a_0^{1-q} - a_{[\frac{\ell}{2}]}^{1-q} \geq C_1[\frac{\ell}{2}].$$

Therefore it follows that

$$\int_{\frac{\ell}{2}} |\nabla(u_\ell - u_\infty)|^q = a_0 \leq \frac{1}{(C_1[\frac{\ell}{2}])^{\frac{1}{q-1}}} \leq \frac{C_2}{\ell^{\frac{1}{q-1}}}.$$

4. PROOF OF THEOREM 1.4 AND SOME ADDITIONAL RESULTS

We do not restrict ourself to the assumption that $p = 1$. We assume that ω_1 is open and bounded subset of \mathbb{R}^p , which is star shaped around the origin.

Lemma 4.1. *Under the assumptions $q = 2$, (1.1) and (1.8), one has $F \in C^1(\mathbb{R}^n)$.*

Proof. Using the assumptions (1.1) and (1.8), it is easy to show approximating $\Delta\phi$ by its discrete expression that

$$0 \leq \langle \Delta F, \phi \rangle \leq 4n\beta \int_{\mathbb{R}^n} \phi, \quad \forall \phi \in C_c^\infty(\mathbb{R}^n), \quad \phi \geq 0.$$

This implies that ΔF belongs to the dual space of $L^1(\mathbb{R}^n)$ and hence to $L^\infty(\mathbb{R}^n)$. Now the lemma follows from the estimates for Newtonian potential (see Lemma 4.1 of [17]). \square

Now we proceed to the proof of Theorem 1.4.

Proof of Theorem 1.4 We have $E_{\Omega_\ell}(u_\ell) \leq E_{\Omega_\ell}(0) = 0$, which implies that

$$\lambda \int_{\Omega_\ell} |\nabla u_\ell|^2 \leq \int_{\Omega_\ell} F(\nabla u_\ell) \leq \int_{\Omega_\ell} f u_\ell.$$

Using Hölder's inequality and then Poincaré's inequality, we have for some constant $C > 0$,

$$(4.1) \quad \int_{\Omega_\ell} |\nabla u_\ell|^2 \leq C \ell^p.$$

For $\ell_0 < \ell - 1$, choose $\rho_{\ell_0} = \rho_{\ell_0}(X_1)$ satisfying $\rho_{\ell_0} = 1$ on $\ell_0\omega_1$ and 0 outside Ω_{ℓ_0+1} . Also assume $|\nabla_{X_1} \rho_{\ell_0}| \leq C$, for some $C > 0$.

Then one can proceed exactly as in the proof of Theorem 1.3, until the inequality

$$\int_{\Omega_{\ell_0+1}} F(\nabla u_\ell) + F(\nabla u_\infty) \leq \int_{\Omega_{\ell_0+1}} F(\nabla \hat{\psi}_{\ell_0,\ell}) + F(\nabla \check{\psi}_{\ell_0,\ell}).$$

Using the convexity of q -type and (1.8) we arrive to

$$\alpha \int_{\Omega_{\ell_0+1}} |\nabla(u_\ell - u_\infty)|^2 \leq \beta \int_{\Omega_{\ell_0+1}} |\nabla(\hat{\psi}_{\ell_0,\ell} - \check{\psi}_{\ell_0,\ell})|^2.$$

Now since $\hat{\psi}_{\ell_0,\ell} - \check{\psi}_{\ell_0,\ell} = (1 - \rho_{\ell_0})(u_\ell - u_\infty)$ and $\rho_{\ell_0} = 1$ on Ω_{ℓ_0} , this implies that

$$\alpha \int_{\Omega_{\ell_0+1}} |\nabla(u_\ell - u_\infty)|^2 \leq \beta \int_{D_{\ell_0}} |\nabla(\hat{\psi}_{\ell_0,\ell} - \check{\psi}_{\ell_0,\ell})|^2.$$

Using (3.17) one obtains for some constant $C = C(\lambda, \lambda_1, \Lambda, \alpha, \beta)$

$$\int_{\Omega_{\ell_0}} |\nabla(u_\ell - u_\infty)|^2 \leq C \int_{D_{\ell_0}} |\nabla(u_\ell - u_\infty)|^2,$$

which is equivalent to

$$(4.2) \quad \int_{\Omega_{\ell_0}} |\nabla(u_\ell - u_\infty)|^2 \leq \frac{C}{C+1} \int_{\Omega_{\ell_0+1}} |\nabla(u_\ell - u_\infty)|^2.$$

Choosing $\ell_0 = \frac{\ell}{2} + m$ for $m = 0, 1, \dots, [\frac{\ell}{2}] - 1$ and iterating (4.2), we get

$$\int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - u_\infty)|^2 \leq \left(\frac{C}{C+1} \right)^{[\frac{\ell}{2}]-1} \int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2.$$

Finally setting $r = \frac{C}{C+1} < 1$ and from (4.1), we have

$$\int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - u_\infty)|^2 \leq C \ell^p e^{([\frac{\ell}{2}]-1) \log r}.$$

Since $\log r < 0$, the theorem follows.

The next proposition gives a sufficient criterion for (1.8) to hold true. More precisely we have :

Proposition 4.1. *If $F \in C^1(\mathbb{R}^n)$ is such that for some $\alpha, \beta > 0$*

$$\alpha |\xi - \eta|^q \leq (\nabla F(\xi) - \nabla F(\eta)) \cdot (\xi - \eta) \quad \forall \xi, \eta \in \mathbb{R}^n$$

or

$$(\nabla F(\xi) - \nabla F(\eta)) \cdot (\xi - \eta) \leq \beta |\xi - \eta|^2, \quad \forall \xi, \eta \in \mathbb{R}^n$$

is satisfied, then the condition (1.1) or respectively (1.8) holds.

Proof. One has

$$\begin{aligned} F(\xi) - F\left(\frac{\xi + \eta}{2}\right) &= - \int_0^1 \frac{d}{dt} F\left(\xi + t\left(\frac{\eta - \xi}{2}\right)\right) dt \\ &= - \frac{1}{2} \int_0^1 \nabla F\left(\xi + t\left(\frac{\eta - \xi}{2}\right)\right) \cdot (\eta - \xi) dt. \end{aligned}$$

Exchanging the roles of ξ and η we get

$$\begin{aligned} F(\eta) - F\left(\frac{\xi + \eta}{2}\right) &= - \int_0^1 \frac{d}{dt} F\left(\eta + t\left(\frac{\xi - \eta}{2}\right)\right) dt \\ &= - \frac{1}{2} \int_0^1 \nabla F\left(\eta + t\left(\frac{\xi - \eta}{2}\right)\right) \cdot (\xi - \eta) dt. \end{aligned}$$

Then adding the two equalities above we obtain

$$\begin{aligned} F(\xi) + F(\eta) - 2F\left(\frac{\xi + \eta}{2}\right) &= \frac{1}{2} \int_0^1 (\nabla F(\eta + t(\frac{\xi - \eta}{2})) - \nabla F(\xi + t(\frac{\eta - \xi}{2}))) \cdot (\eta - \xi) dt \end{aligned}$$

Noting that

$$[\eta + t(\frac{\xi - \eta}{2})] - [\xi + t(\frac{\eta - \xi}{2})] = (1 - t)(\eta - \xi)$$

we obtain from our assumptions

$$\frac{\alpha}{2} \int_0^1 (1-t)^{q-1} |\eta - \xi|^q dt \leq F(\xi) + F(\eta) - 2F\left(\frac{\xi + \eta}{2}\right)$$

or

$$F(\xi) + F(\eta) - 2F\left(\frac{\xi + \eta}{2}\right) \leq \frac{\beta}{2} \int_0^1 (1-t)^{q-1} |\eta - \xi|^q dt$$

i.e.

$$\frac{\alpha}{2q} |\eta - \xi|^q \leq F(\xi) + F(\eta) - 2F\left(\frac{\xi + \eta}{2}\right)$$

or

$$F(\xi) + F(\eta) - 2F\left(\frac{\xi + \eta}{2}\right) \leq \frac{\beta}{2q} |\eta - \xi|^q \quad \text{for } q = 2.$$

This completes the proof of the proposition. \square

Our main result Theorem 1.3 works only in the case when $p = 1$. Next we provide some partial result (Theorem 4.1) in the case when $0 < p < n$ and $\Omega_\ell = (-\ell, \ell)^p \times \omega_2$ are hypercubes.

Lemma 4.2. [A pointwise estimate] *If $f \geq 0$, then $0 \leq u_\ell \leq u_\infty$ a.e. for all ℓ .*

Proof. Fix $\ell > 0$. First we claim that $u_\ell, u_\infty \geq 0$. We will prove the claim only for u_ℓ , since the proof for u_∞ is identical. Define the function $w_\ell = \max\{0, u_\ell\}$. Clearly $w_\ell \in W_0^{1,q}(\Omega_\ell)$ is non negative and since $f \geq 0$, we have

$$E_{\Omega_\ell}(w_\ell) \leq E_{\Omega_\ell}(u_\ell).$$

The claim then follows from the uniqueness of u_ℓ . Set

$$\mathcal{A}_\ell := \{X \in \Omega_\ell \mid u_\ell(X) > u_\infty(X_2)\}.$$

We claim that

$$E_{\mathcal{A}_\ell}(u_\ell) \leq E_{\mathcal{A}_\ell}(u_\infty).$$

Indeed, if not, setting $v_\ell = u_\ell - (u_\ell - u_\infty)^+$ one has $v_\ell \in W_0^{1,q}(\Omega_\ell)$ and

$$\begin{aligned} E_{\Omega_\ell}(v_\ell) &= E_{\mathcal{A}_\ell}(v_\ell) + E_{\Omega_\ell \setminus \mathcal{A}_\ell}(v_\ell) = E_{\mathcal{A}_\ell}(u_\infty) + E_{\Omega_\ell \setminus \mathcal{A}_\ell}(u_\ell) \\ &< E_{\mathcal{A}_\ell}(u_\ell) + E_{\Omega_\ell \setminus \mathcal{A}_\ell}(u_\ell) = E_{\Omega_\ell}(u_\ell) \end{aligned}$$

and a contradiction with the definition of u_ℓ . Setting then $w_\ell = u_\infty + (u_\ell - u_\infty)^+$ one has $w_\ell \in V^{1,q}(\Omega_\ell)$ and

$$E_{\Omega_\ell}(w_\ell) = E_{\mathcal{A}_\ell}(w_\ell) + E_{\Omega_\ell \setminus \mathcal{A}_\ell}(w_\ell) = E_{\mathcal{A}_\ell}(u_\ell) + E_{\Omega_\ell \setminus \mathcal{A}_\ell}(u_\infty) \leq E_{\Omega_\ell}(u_\infty).$$

Thus $w_\ell = u_\infty$ and $(u_\ell - u_\infty)^+ = 0$ which completes the proof. \square

Using similar argument as in the last theorem one can prove the following monotonicity property of the solutions u_ℓ . We emphasize that such a monotonicity property holds true for general domains, but we will present the result only for the family Ω_ℓ .

Lemma 4.3. [Monotonicity] *If $f \geq 0$ and $\ell < \ell'$ then $u_\ell \leq u_{\ell'}$, a.e. in $\Omega_{\ell'}$, where u_ℓ is extended by 0 outside Ω_ℓ .*

Theorem 4.1. *Let $f \geq 0$, then for some function $\tilde{u}_\infty = \tilde{u}_\infty(X_2)$ it holds*

$$u_\ell \rightarrow \tilde{u}_\infty(X_2)$$

pointwise.

Proof. In the statement of the theorem it is understood that u_ℓ are extended by 0 outside Ω_ℓ . For $h \in \mathbb{R}$, we set

$$\tau_h^i v := v(X - he_i) \quad i = 1, \dots, p$$

where e_i denotes the unit vector in i -th direction. First we claim that

$$(4.3) \quad u_{\ell+h} \geq \tau_h^i u_\ell.$$

In order to prove (4.3) one has to show that $(\tau_h^i u_\ell - u_{\ell+h})^+ = 0$. Set

$$\mathcal{A} := \{X \in \Omega_{\ell+h} \mid \tau_h^i u_\ell(X) > u_{\ell+h}(X)\}.$$

We have then $E_{\mathcal{A}}(u_{\ell+h}) \leq E_{\mathcal{A}}(\tau_h^i u_\ell)$. Indeed if this is not true, then setting

$$v := u_{\ell+h} + (\tau_h^i u_\ell - u_{\ell+h})^+$$

one has $v \in W_0^{1,q}(\Omega_{\ell+h})$ and

$$\begin{aligned} E_{\Omega_{\ell+h}}(v) &= E_{\mathcal{A}}(\tau_h^i u_\ell) + E_{\Omega_{\ell+h} \setminus \mathcal{A}}(u_{\ell+h}) \\ &< E_{\mathcal{A}}(u_{\ell+h}) + E_{\Omega_{\ell+h} \setminus \mathcal{A}}(u_{\ell+h}) = E_{\Omega_{\ell+h}}(u_{\ell+h}) \end{aligned}$$

and a contradiction with the definition of $u_{\ell+h}$.

Define $\mathcal{A}' := \{X \in \Omega_\ell \mid \tau_{-h}^i(u_{\ell+h})(X) < u_\ell(X)\}$. We claim that

$$(4.4) \quad \mathcal{A} = \mathcal{A}' + he_i.$$

Indeed for X such that $x_i < -\ell + h$ one has $\tau_h^i u_\ell = 0$ that is X does not belongs to \mathcal{A} and for $x_i - h \geq -\ell$,

$$\tau_h^i(u_\ell(X)) > u_{\ell+h}(X) \iff u_\ell(X - he_i) > u_{\ell+h}(X),$$

i.e. $X - he_i \in \mathcal{A}'$ which proves (4.4).

We consider then $w = u_\ell - (u_\ell - \tau_{-h}^i u_{\ell+h})^+ \in W_0^{1,q}(\Omega_\ell)$. Clearly the function vanishes when $u_\ell = 0$ since $u_{\ell+h} \geq 0$. Then one has

$$\begin{aligned} E_{\Omega_\ell}(w) &= E_{\Omega \setminus \mathcal{A}'}(u_\ell) + E_{\mathcal{A}'}(\tau_{-h}^i u_{\ell+h}) \\ &= E_{\Omega \setminus \mathcal{A}'}(u_\ell) + \int_{\mathcal{A}'} F(\nabla u_{\ell+h}(X + he_i)) - f u_{\ell+h}(X + he_i) \\ &= E_{\Omega \setminus \mathcal{A}'}(u_\ell) + \int_{\mathcal{A}} F(\nabla u_{\ell+h}(X)) - f u_{\ell+h}(X) \\ &= E_{\mathcal{A}}(u_{\ell+h}) + E_{\Omega \setminus \mathcal{A}'}(u_\ell) \leq E_{\mathcal{A}}(\tau_h^i u_\ell) + E_{\Omega \setminus \mathcal{A}'}(u_\ell) \\ &= E_{\mathcal{A}'}(u_\ell) + E_{\Omega \setminus \mathcal{A}'}(u_\ell) = E_{\Omega_\ell}(u_\ell). \end{aligned}$$

By the definition and uniqueness of u_ℓ this implies that

$$w = u_\ell \iff (u_\ell - \tau_{-h}^i u_{\ell+h})^+ = 0$$

and $\mathcal{A}', \mathcal{A}$ are of measure 0. This proves (4.3). With similar argument one can show that

$$(4.5) \quad u_{\ell+h} \geq \tau_{-h}^i u_\ell.$$

Since u_ℓ is a monotone increasing sequence of functions which are bounded above by $u_\infty(X_2)$ (from Lemma 4.2 and Lemma 4.3), one has for some \tilde{u}_∞ ,

$$u_\ell \rightarrow \tilde{u}_\infty$$

pointwise. From (4.5) it follows that

$$\tilde{u}_\infty(X) \geq \tilde{u}_\infty(X - he_i), \quad \tilde{u}_\infty(X) \geq \tilde{u}_\infty(X + he_i) \quad \forall h, i \in \{1, \dots, p\}$$

and thus \tilde{u}_∞ is independent of the variable X_1 . \square

We would like to point out here that it is possible to show that $\tilde{u}_\infty = u_\infty$. We refer to M. Chipot [3].

Now we are interested in asymptotic behavior of the sequence $\frac{E_{\Omega_\ell}(u_\ell)}{|\ell\omega_1|}$ as $\ell \rightarrow \infty$. ($|\cdot|$ denotes the measure of a set). In particular we will prove the following theorem.

Theorem 4.2. [Convergence of energy] *One has for some constant $C > 0$ and sufficiently large ℓ ,*

$$E_{\omega_2}(u_\infty) \leq \frac{E_{\Omega_\ell}(u_\ell)}{|\ell\omega_1|} \leq E_{\omega_2}(u_\infty) + \frac{C}{\ell}$$

where u_ℓ and u_∞ as in (1.3) and (1.5) respectively.

Proof. Set

$$v_\ell(X_2) = \int_{\ell\omega_1} u_\ell(\cdot, X_2) = \frac{1}{|\ell\omega_1|} \int_{\ell\omega_1} u_\ell(\cdot, X_2).$$

It is easy to see that $v_\ell \in W_0^{1,q}(\omega_2)$. Therefore

$$E_{\omega_2}(u_\infty) \leq E_{\omega_2}(v_\ell) = \int_{\omega_2} F(0, \nabla_{X_2} v_\ell) - f v_\ell.$$

From divergence theorem, one has

$$0 = \int_{\ell\omega_1} \nabla_{X_1} u_\ell$$

and by differentiation under the integral

$$\nabla_{X_2} v_\ell = \int_{\ell\omega_1} \nabla_{X_2} u_\ell.$$

Therefore we have by Jensen's inequality

$$E_{\omega_2}(u_\infty) \leq \int_{\omega_2} \left\{ F \left(\int_{\ell\omega_1} \nabla_{X_1} u_\ell, \int_{\ell\omega_1} \nabla_{X_2} u_\ell \right) \right\} - \int_{\omega_2} f \left\{ \int_{\ell\omega_1} u_\ell \right\}$$

$$\leq \int_{\omega_2} \int_{\ell\omega_1} \{F(\nabla u_\ell) - f u_\ell\} = \frac{E_{\Omega_\ell}(u_\ell)}{|\ell\omega_1|}.$$

This proves the first inequality.

For the second one, first we consider a Lipschitz continuous function $\rho_\ell = \rho_\ell(X_1)$, such that $\rho_\ell = 1$ on $(\ell-1)\omega_1$ and $\rho_\ell = 0$ on $\partial(\ell\omega_1)$. We also assume that there exists a constant $C > 0$ (independent of ℓ) such that

$$|\nabla_{X_1} \rho_\ell| \leq C \text{ and } 0 \leq \rho_\ell \leq 1.$$

Thus $\rho_\ell u_\infty \in W_0^{1,q}(\Omega_\ell)$. Then from (1.3), we have

$$E_{\Omega_\ell}(u_\ell) \leq E_{\Omega_\ell}(\rho_\ell u_\infty).$$

We compute

$$\begin{aligned} E_{\Omega_\ell}(\rho_\ell u_\infty) &= E_{\Omega_{\ell-1}}(u_\infty) + \int_{\Omega_\ell \setminus \Omega_{\ell-1}} F(\nabla(\rho_\ell u_\infty)) - f u_\infty \rho_\ell \\ &\leq E_{\Omega_\ell}(u_\infty) + \int_{\Omega_\ell \setminus \Omega_{\ell-1}} F(\nabla(\rho_\ell u_\infty)) - F(\nabla u_\infty) - f u_\infty (\rho_\ell - 1) \\ &\leq |\ell\omega_1| E_{\omega_2}(u_\infty) + \int_{\Omega_\ell \setminus \Omega_{\ell-1}} \Lambda |\nabla(\rho_\ell u_\infty)|^q + \Lambda |\nabla u_\infty|^q + |f| |u_\infty| \\ &\leq |\ell\omega_1| E_{\omega_2}(u_\infty) + C \int_{\Omega_\ell \setminus \Omega_{\ell-1}} |\nabla u_\infty|^q + |u_\infty|^q + |f|^{q'} \\ &\leq |\ell\omega_1| E_{\omega_2}(u_\infty) + C \{|\ell\omega_1| - |(\ell-1)\omega_1|\} \int_{\omega_2} |\nabla u_\infty|^q + |u_\infty|^q + |f|^{q'}. \end{aligned}$$

Dividing by $|\ell\omega_1|$ and the result follows, i.e. one has

$$\frac{E_{\Omega_\ell}(u_\ell)}{|\ell\omega_1|} \leq E_{\omega_2}(u_\infty) + \frac{C}{\ell}.$$

This finishes the proof of the theorem. \square

Acknowledgment The authors of this paper would like to thank the anonymous referee for his/her valuable suggestions in improving the quality of this work. The argument from (3.20) till the end of the proof of our main theorem is due to him which provided a better rate of convergence. The research leading to these results has received funding from Lithuanian-Swiss cooperation programme to reduce economic and social disparities within the enlarged European Union under project agreement No CH-3-SMM-01/0. The research of the first author was supported also by the Swiss National Science Foundation under the contracts # 200021-129807/1 and 200021-146620. The research of the second author was supported by the European Initial Training Network FIRST under the grant agreement # PITN-GA-2009-238702. The research of the third author is funded by “Innovation in Science Pursuit for Inspired Research(INSPIRE)” under the IVR Number: 20140000099.

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(Michel Chipot)
INSTITUTE FÜR MATHEMATIK, UNIVERSITÄT ZÜRICH,
WINTERTHURERSTR. 190, CH-8057 ZÜRICH, SWITZERLAND.
E-mail address: `m.m.chipot@math.uzh.ch`

(Aleksandar Mojsic)
HELMUT SCHMIDT UNIVERSITY / UNIVERSITY OF THE FEDERAL ARMED FORCES HAMBURG,
DEPARTMENT OF MECHANICAL ENGINEERING,
HOLSTENHOFWEG 85
22043 HAMBURG, GERMANY.
E-mail address: `mojsica@hsu-hh.de`

(Prosenjit Roy)
TATA INSTITUTE OF FUNDAMENTAL RESEARCH
SHARADANAGAR, GKVK CAMPUS, POSTBOX - 560065,
BANGALORE, INDIA .
E-mail address: `prosenjit@math.tifrbng.res.in`